Dynamic Model of Losses of Creditor with a Large Mortgage Portfolio

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Abstract. We propose a dynamic model of mortgage credit losses. We assume borrowers to hold assets covering the instalments and to own a real estate which serves as a collateral; both the value of the assets and the price of the estate follow general stochastic processes driven by common and individual factors. We describe the correspondence between the common factors, the percentage of defaults and the loss given default and we suggest a procedure of econometric estimation of the model.

Keywords: credit risk, mortgage, loan portfolio, dynamic model, estimation

JEL classification: G32

1. Introduction

One of the sources of the recent financial crisis was the collapse of the mortgage business. Even if there are there are ongoing disputes about the causes of the collapse, wrong risk management seems to be one of them. Hence, realistic models of the lending institutions' risk are of a great importance.

The textbook approach to the risk control of the loans' portfolio, which is also a part of the standard Basel II (2006), is that of Vasicek (2002) who deduces the rates of defaults of the borrowers, and consequently the losses of the banks, from the value of the borrowers' assets following a geometric Brownian motion.

We generalize the model in three ways:

1. We add a dynamics to the model (note that the Vasicek's model is only one-period one).
2. We allow more general distribution of the assets.
3. We add a sub-model of the losses given default which allows us to calculate the overall percentage loss of the bank.

Similarly to the paper we generalize, there is a one-to-one correspondence between the common factors and the percentage of defaults (PD) and losses given default (LGD) in our model; using this, an econometrics of the bivariate series of PD's and LGD's may be done.

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As to our knowledge, no dynamic generalization of the Vasicek's model incorporating the losses given default has been published yet. However, our approach to the dynamics and/or common modelling of PDs and LGDs is not the only one:

- There are other, maybe more exact, ways to get the relevant information from the past of the system, e.g. credit scoring (see Vasicek (2002) where the distribution of the losses is a function of the probability of default) or observing the credit derivatives (see d'Ecclesia (2008)). Another approach to the dynamics of could be to track the situation of individual client's (see e.g. Gupton at. al. (1997)) The usefulness of our approach, however, could lie in the fact that our model is applicable "from outside" (in the sense that it does not require banks' internal information).
- Neither the joint modelling of the PD and the LGD is our exclusive invention (see e.g. Witzany (2010) and the references therein); the novelty of our approach, however, is the fact that the form of the dependence of the LGD on the common factor arises naturally from the matter of fact rather than from any ad hoc assumption in our model.

We do not perform actual estimations in the present paper; instead, we refer the reader to papers Gapko and Šmíd (2010a) and Gapko and Šmíd (2010b) in which slightly simpler models of PD's, LGD's respectively, are applied to real data.

The paper is organized as follows: after the general definitions (Section 2), the models of PD's (Section 3) and LGD's (Section 4) are constructed. Then (Section 5) a procedure of econometric estimation of the model is outlined. Finally (Section 6), the paper is concluded.

2. The Setting

Let there be countably many potential borrowers, the i-th of them owning assets with value A_t^i, $t \geq 0$. At a time $S^i \in \mathbb{N}_0$, the i-th borrower takes a mortgage of amount C^i with help of which he buys a real property with price dC^i, $d \geq 1$. The mortgage is repaid by instalments amounting to bC^i, $b > 0$, at each of the times $S^i + 1$, $S^i + 2$, $\dots$, $S^i + r$ where $r \in \mathbb{N}$ is the same for all the borrowers.

Quite realistically, we assume that the value of the assets of the borrower is, in some sense, proportional to the size of the mortgage: in particular,

$$a_{t-}^i = a_{t-1}^i \exp\left\{\Delta\tilde{Y}_t + \Delta\tilde{Z}_t^i\right\}. \qquad t \in \mathbb{N},\, t \geq S^i,$$

where $a_t^i = A_t^i / C^i$ is the value of the assets per the unit of the debt, $\tilde{Y}_t$ is a common factor (e.g. stock index) and $\tilde{Z}_t^i$, $\mathbb{E}\Delta\tilde{Z}_t^i = 0$, is an individual factor such that $(\Delta\tilde{Z}_t^i)_{i\in\mathbb{N},t>0}$ are i.i.d. and independent of $(\Delta\tilde{Y}_t)_{t>0}$ and that the c.d.f. $\tilde{\psi}$ of ΔZ_1^1 is continuous, strictly increasing on $\mathbb{R}$ (Δ stands for a one-period difference).

At each period, there is a fixed percentage $1/r$ of new mortgages in the portfolio. We greatly simplify our future calculations by assuming that, for any newly coming debtor i, the conditional distribution of a_t^i given ω_t is the same as that of the old debtors. Even if this assumption may seem deliberate, it is not in fact, because banks give a loan only to solvent clients (i.e. those who prove that they would have been able to pay the instalments in the past as well). If we, in addition, suppose the client to spend the potential instalments for something else in the past (e.g. for saving), we get exactly the same financial history of a client whether he pays a mortgage or not.

We assume that the instalments are paid by means of selling the necessary amount of the assets, i.e.

$$a_t^i = a_{t-}^i - b, \qquad t \in \mathbb{N}, \, t > S^i.$$

If $a_t^i < 0$ then we say that the borrower defaults at t.

Further, in our model, the price P_t^i of the real property, ownded by the i-th debtor, fulfils

$$P_{S^i}^i = dC^i, \qquad P_t^i = \exp\left\{\Delta I_t + \Delta E_t^i\right\} P_{t-1}^i, \quad t > S^i,$$

where I_t is another common factor (e.g. real estate price index) and E_t^i, $\mathbb{E}\Delta E_t^i = 0$, is an individual factor such that $(\Delta E_t^i)_{i\in\mathbb{N},t>0}$ are i.i.d., independent of $(\Delta\tilde{Z}_t^i)_{i,t>0}, \tilde{Y}, I$. Again we assume the c.d.f ϕ of E_1^1 to be continuous strictly increasing.

3. The Portfolio of Loans

Fix $t \in \mathbb{N}$ and renumber the potential borrowers so that only those who are active since $t-1$ to t (i.e. those with $t-r \le S^i \le t-1$) and did not default until $t-1$ are numbered.

Introduce Q^i as a zero-one variable indicating whether the i-th borrower defaults at t:

$$Q^i = \mathbf{1}[a_{t-}^i < b] = \mathbf{1}[\dot{a}_{t-1}^i + \Delta\tilde{Y}_t + \Delta Z_t^i < \dot{b}]$$

where $\dot{x}$ stands for $\log x$ for any number x here as well as in the sequel. Without loss of generality, we incorporate the relative instalments into the common factor and normalize the equation by $\sigma = \sqrt{\mathrm{var}\Delta\tilde{\mathrm{Z}}_1^1}$:

$$Q^i = \mathbf{1}[\ddot{a}_{t-1}^i + \Delta Y_t + \Delta Z_t^i < 0]$$

where $\Delta Y_t = (\Delta\tilde{Y}_t - \dot{b})/\sigma$, $\Delta Z_t^i = \Delta\tilde{Z}_t^i/\sigma$, $\ddot{a}_{t-1}^i = \dot{a}_{t-1}^i/\sigma$.

To simplify the notation, denote

$$\omega_t = (Y_1, I_1, Y_2, I_2, \ldots, Y_t, I_t)$$

the history both the common factors up to t,

$$\Omega_t = (Y_\tau, I_\tau, Z_\tau^1, E_\tau^1, Z_\tau^2, E_\tau^2, \ldots)_{\tau \ge t}$$

the future of all the factor starting from t and

$$\Omega_0 = (\ddot{a}_0^1, \ddot{a}_1^1, \ldots, \Omega_1).$$

Assume that the (conditional) distribution of C^i is the same for all the borrowers and that it depends only on the time of the start of the mortgage and on the common factors. Reformulated an exact way:

C For any i, C^i is conditionally independent of $\Omega_0, C^1, \ldots, C^{i-1}, C^{i+1}, \ldots$ given (S^i, ω_{S^i}) and the conditional distribution of C^i given (S^i, ω_{S^i}) equals for all i.

Further, assume

I(t) $S^1, S^2, \dots$ are mutually independent, independent of Ω_0 and $\mathbb{P}[S^i = s] = \frac{1}{r}$, $s = t-r, \dots, t-1$, $i \in \mathbb{N}$.

and denote
$$Q_t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Q^i$$
the percentage of the debtors who defaulted at t.[1] Given that

A(t) $\ddot{a}^1_{t-1}, \ddot{a}^2_{t-1}, \dots$ are mutually conditionally independent given ω_{t-1}, and conditionally independent of Ω_t given ω_{t-1} with a common strictly increasing continuous conditional c.d.f. ϑ_{t-1} given ω_{t-1}

we get, similarly to Vasicek (2002), by the Law of Large Numbers applied to a conditional distribution, that
$$Q_t = \mathbb{E}(Q^1|Y_t, \omega_{t-1}) = \mathbb{P}[\ddot{a}^1_{t-1} + \Delta Y_t + \Delta Z^1_t < 0|Y_t, \omega_{t-1}] = \Psi_t(-\Delta Y_t) \tag{1}$$
where Ψ_t is the conditional c.d.f. of $\ddot{a}^i_{t-1} + \Delta Z^i_t$ given ω_{t-1} (i.e. the c.d.f. of the convolution of ϑ_{t-1} and ψ where ψ is the c.d.f. of ΔZ^1_1). Therefore,

Lemma 1. *For given ω_{t-1}, there exists a one to one mapping between Y_t and Q_t given by (1). Specially,*
$$\Delta Y_t = -\Psi_t^{-1}(Q_t). \tag{2}$$

Proof. The Lemma follows from (1) and from the measurability of ϑ_{t-1} with respect to ω_{t-1}. □

Before we go on, let us point out that the default of a borrower depends neither on the "age" of the mortgage nor on the other defaults.

Lemma 2. $S^1, Q^1, S^2, Q^2 \dots$ *are mutually conditionally independent given* ω_t

Proof. By I(t), the variables $S_1, S_2, \dots$ are mutually independent and independent of the variables defining $Q^\bullet$. By A(t) and the mutual independence of the individual factors, the variables $Q^1, Q^2, \dots$ are mutually conditionally independent given ω_t. The Lemma follows from the chain rule Kallenberg (2002), Proposition 6.8. □

Now, let us proceed to the portfolio at the next period: First, let us exclude all the borrowers who defaulted at t (i.e. with $Q^i = 1$) and denote $[j]$ the index of the j-th borrower out of those who did not default at t. Thanks to the independence of S^i of the variables underlying Q^i, all the "ages" occur uniformly among the "survivors":

Lemma 3. $S^{[1]}, S^{[2]}, \dots$ *are mutually independent, independent of* Ω_0 *and* $\mathbb{P}[S^{[j]} = s] = 1/r$ *for any* $t - r \le s \le t-1$ *and* $j \in \mathbb{N}$.

[1]This and all the other cases of the convergence are in probability.

Proof. By I(t), all the random variables $[1], [2], \dots$ are defined by means of variables, independent of $S^1, S^2, \dots$ hence, the Lemma might follows from Hoffmann-Jørgenson (1994) 4.5.2. □

As to the assets of the survivors, we are getting

Lemma 4. $\ddot{a}_t^{[1]}, \ddot{a}_t^{[2]}, \dots$ *are conditionally i.i.d. given* ω_t, *conditionally independent of* Ω_{t+1} *given* ω_t, *and*

$$\mathbb{P}\left[a_t^{[1]} \le z\right] = \frac{\Psi_t(z - \Delta Y_t) - Q_t}{1 - Q_t}, \qquad z \ge 0.$$

Proof. The conditional independence follows from the (conditional) independence of the underlying variables. The formula on the RHS is that of the distribution of the debtor's assets immediately before t, truncated at zero. A formal proof may be found in Šmíd (2010). □

Now let us remove all the debtors, who successfully finished paying the mortgage at t, from the portfolio and add the newcomers. Since the both the ratios of the leavers and the newcomers is $1/r$ according to Lemma 3, our assumptions respectively, and since we supposed the initial distribution of the assets to be the same for the newcomers as for the active debtors, we may, without any loss of generality, give the indices of the leavers to the newcomers; together with Lemma 3, this will clearly imply I(t+1) with $[i]$ instead of i. Moreover, it follows from Lemma 4, that A(t+1) holds true with $[i]$ instead of i and with

$$\vartheta_{t-1}(z) = \frac{\Psi_{t-1}(z - \Delta Y_{t-1}) - Q_{t-1}}{1 - Q_{t-1}}. \tag{3}$$

Summarized,

Corollary 1. *C+I(1)+A(1) implies I(t)+A(t) for any* t *with (3) given a subsequent renumbering by* $[\dots]$ *for any* $t > 1$.

Until the end of the paper, assume C+I(1)+A(1) and note that the conditional independence in A(1) is in fact the ordinary one.

4. The Loss of the Bank

In the present Section, we shall study the (percentage) loss of the bank at the time t: Denote H^i the exposure at default (i.e. the remaining debt) of the i-th borrower at t and assume that

$$H^i = p(t - S^i)C^i,$$

for some decreasing function fulfilling $p(1) = 1$ (the shape of p may depend on the way of interest calculation and the accounting rules of the bank). The amount which the bank will recover in case of the default of the i-th debtor at time t is then

$$\begin{aligned} G^i = \min(P_t^i, H^i) &= C^i \min\left(d \exp\left\{\sum_{i=S^i+1}^{t} \left[\Delta I_t^i + \Delta E_t^i\right]\right\}, p(t - S^i)\right) \\ &= C_i^i \exp\left\{\min\left(\dot{d} + \sum_{i=S^i+1}^{t} \left[\Delta I_t^i + \Delta E_t^i\right]\right), \log(p(t - S^i))\right\}. \end{aligned}$$

while the percentage loss given default, i.e. the ratio of the actual losses and the total exposure at default, comes out as

$$\begin{aligned} L_t &= \lim_{n\to\infty} \frac{\sum_{i=1}^n Q^i(H^i - G^i)}{\sum_{i=1}^n Q^i H^i} = 1 - \lim_{n\to\infty} \frac{n^{-1}\sum_{i=1}^n Q^i G^i}{n^{-1}\sum_{i=1}^n Q^i H^i} \\ &= 1 - \frac{\lim_{n\to\infty} n^{-1}\sum_{i=1}^n Q^i G^i}{\lim_{n\to\infty} n^{-1}\sum_{i=1}^n Q^i H^i} \end{aligned}$$

(the last identity holds thanks to Kallenberg (2002), Corollary 4.5.).

It follows from our assumptions that, in both the sums, the summands are conditionally i.i.d. given ω_t, hence

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^n Q^i G^i = \mathbb{E}(Q^1 G^1|\omega_t) = \mathbb{E}(Q^1|\omega_t)\mathbb{E}(G^1|\omega_t) = Q_t\mathbb{E}(G^1|\omega_t)$$

by the LLN (first "="), Lemma 2 together with assumptions C and I(t) (second "=") and relation (1) (third "="). Since, analogously,

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^n Q^i H^i = Q_t\mathbb{E}(H^1|\omega_t),$$

we are getting

$$L_t = 1 - \frac{\mathbb{E}(G^1|\omega_t)}{\mathbb{E}(H^1|\omega_t)}.$$

Evaluating both the expectations, we get

$$\mathbb{E}(H^1|\omega_t) = \mathbb{E}\left(\mathbb{E}(H^1|\omega_t, S^1)|\omega_t\right) = \sum_{s=t-r}^{t-1} \frac{1}{r} p(t-s)c_s(\omega_{t-1}),$$

$$c_s(\omega) = \mathbb{E}(C^1|S^1 = s, \omega_{t-1} = \omega)$$

and

$$\mathbb{E}(G^1|\omega_t) = \mathbb{E}\left(\mathbb{E}(G^1|\omega_t, S^1)|\omega_t\right) = \sum_{s=t-r}^{t-1} \frac{1}{r} p(t-s)c_s(\omega_{t-1})e_s(\tilde{I}_s)$$

$$e_s(\iota) = d\mathbb{E}\left(\exp\left\{\min\left(\tilde{I}_s + \tilde{E}_s, w_s\right)\right\} |\tilde{I}_s = \iota\right)$$

where $\tilde{I}_s = I_t - I_s$, $\tilde{E}_s = E_t^1 - E_s^1$ and $w_s = \log(p(t-s)) - \dot{d}$. Further, we are getting

$$\begin{aligned} e_s(\iota) &= d\mathbb{E}(\exp\left\{\tilde{I}_s\right\} \exp\left\{\min(\tilde{E}_s, w_s - \tilde{I}_s)\right\} |\tilde{I}_s = \iota) \\ &= de^\iota \left[\int_{-\infty}^{w_s-\iota} e^x d\Phi_{(t-s)}(x) + e^{w_s-\iota}(1 - \Phi_{(t-s)}(w_s - \iota))\right] \\ &= de^\iota \int_{-\infty}^{w_s-\iota} e^x d\Phi_{(t-s)}(x) + p(t-s)\left[1 - \Phi_{(t-s)}(w_s - \iota)\right] \end{aligned}$$

where $\Phi_{(\nu)}$ is a c.d.f. of the sum of ν independent copies of ΔE_1^1 (i.e. the $\nu-$th convolution of ϕ). Consequently, the loss given default equals to

$$L_t = 1 - \tfrac{1}{\sum_s p(t-s)c_s(\omega_{t-1})} \sum_{s=t-r}^{t-1} p(t-s)c_s(\omega_{t-1})h_s\left(\tilde{I}_s\right), \tag{4}$$

where

$$h_s(\iota) = de^{\iota} \int_{-\infty}^{w_s-\iota} e^x d\Phi_{(t-s)}(x) + p(t-s)[1-\Phi_{(t-s)}(w_s-\iota)]$$

or, integrating by parts,

$$\begin{aligned} h_s(\iota) &= de^{\iota}\left[\Phi_{(t-s)}(w_s-\iota)e^{w_s-\iota} - \int_{-\infty}^{w_s-\iota} \Phi_{(t-s)} e^x dx\right] + p(t-s)[1-\Phi_{(t-s)}(w_s-\iota)] \\ &= p(t-s) - de^{\iota} \int_{-\infty}^{w_s-\iota} \Phi_{(t-s)}(x) e^x dx. \end{aligned}$$

Since, by the latter formula,

$$\begin{aligned} \frac{\partial}{\partial \iota} h_s(\iota) &= de^{\iota}\left(\Phi_{(t-s)}(w_s-\iota)e^{w_s-\iota} - \int_{-\infty}^{w_s-\iota} \Phi_{(t-s)}(x)e^x dx\right) = \\ &= de^{\iota}\left(\Phi_{(t-s)}(w_s-\iota)\int_{-\infty}^{w_s-\iota} e^x dx - \int_{-\infty}^{w_s-\iota} \Phi_{(t-s)}(x)e^x dx\right) \\ &= de^{\iota}\left(\int_{-\infty}^{w_s-\iota} \Phi_{(t-s)}(w_s-\iota)e^x dx - \int_{-\infty}^{w_s-\iota} \Phi_{(t-s)}(x)e^x dx\right) \\ &= de^{\iota}\left(\int_{-\infty}^{w_s-\iota} \left[\Phi_{(t-s)}(w_s-\iota) - \Phi_{(t-s)}(x)\right] e^x dx\right) > 0 \end{aligned}$$

we see that h is strictly increasing. Consequently,

Lemma 5. *For given ω_{t-1} there is one-to-one mapping between L_t and I_t, given by (4). Specially,*

$$\Delta I_t = \Upsilon_t^{-1}(1-L_t), \tag{5}$$

where

$$\Upsilon_t(\iota) = \frac{1}{\sum_s c_s(\omega_{t-1})} \sum_{s=t-r}^{t-1} \left[c_s(\omega_{t-1}) h_s\left(I_{s+1} + \cdots + I_{t-1} + \iota\right)\right]$$

To close the topic of the losses of the bank, let us note that the *overall* percentage loss of the bank is, given our assumptions,

$$K_t = \lim_{n\to\infty} \frac{\sum_{i=1}^n Q^i(H^i - G^i)}{\sum_{i=1}^n H^i} = Q_t L_t$$

(the RHS is easy to obtain analogously to the previous text).

5. Notes on Econometrics of the Model

Assume we have the sample

$$Q_1, L_1, Q_2, L_2, \ldots, Q_T, L_T$$

at our disposal and want to infer (some of) the parameters of our model, whose list is

$$\mathcal{L}(Y, I),\ c_s(\omega),\ r,\ d,\ p,\ \vartheta_0,\ \psi,\ \phi.$$

Clearly, some further simplification of such a rich parameter space has to be done; this is, however, a task for potential users of the model. Here we only outline a possible procedure of such an inference.

Step 1 – Extraction of Y

Since the expectation and variance of ΔZ_1^1 are given (equal to 0, 1 respectively), it does not seem to be a big harm of reality to assume a particular (say standard normal) distribution of ΔZ_1^1. If we, a bit more arbitrarily, assume that $\ddot{a}_0^i = Y_0 + \epsilon_0^i$ where $\epsilon \sim \mathrm{N}(0, \sigma_0^2)$ for some σ_0 then we get $\Psi_1(z) = \psi\left((z - Y_0)/\sqrt{1+\sigma_0^2}\right)$ and, by (2), $Y_1 = \sqrt{1+\sigma_0^2}\psi^{-1}(Q_1)$. To get $Y_2, Y_3, \dots$, one can use (2) again; however, it is non-trivial to compute the c.d.f.'s $\Psi_2, \Psi_3, \dots$ each of which is a convolution of standard normal distribution with a truncation of an (already non-trivial) distribution; it seems that the only way to evaluate the c.d.f.'s is to use a Monte Carlo simulation.

Step 2 – Extraction of I

Here, the situation is a bit more complicated, because the mapping transforming L to I is more complex and depends on a greater number of the parameters. Therefore, we assume that the values of r, d and the shapes of p and c are known to us. Again we shall suppose the individual factor ΔE_1^1 to be normal; however, since the ratio of the exposure to the individual factor influences the result strongly, we cannot normalize the individual factor here. Even if the variance of ΔE_1^1 could be incorporated into the estimation of the whole model, see Gapko and Šmíd (2010b), here we assume the variance of ΔE_1^1 to be known, equal to $\rho^2 > 0$. Thanks to the normality, we may avoid the integration when computing h_s: if we denote φ the standard normal c.d.f. and put $\varrho = \sqrt{n}\rho$, we get

$$\begin{aligned}
\int_{-\infty}^{w_s-\iota} e^x d\Phi_{(t-s)}(x) &= \int_{-\infty}^{w_s-\iota} \frac{1}{\sqrt{2\pi}\varrho} e^{-\frac{x^2}{2\varrho^2}} e^x dx \\
&= \frac{1}{\sqrt{2\pi}\varrho} \int_{-\infty}^{w_s-\iota} \exp\left\{ -\frac{x^2 - 2\varrho^2 x + \varrho^4}{2\varrho^2} + \frac{1}{2}\varrho^2 \right\} dx \\
&= \exp\left\{\frac{1}{2}\varrho^2\right\} \int_{-\infty}^{w_s-\iota} \frac{1}{\sqrt{2\pi}\varrho} \exp\left\{ -\frac{(x-\varrho^2)^2}{2\varrho^2} \right\} dx \\
&= \exp\left\{\frac{1}{2}\varrho^2\right\} \mathbb{P}\left[N(\varrho^2, \varrho^2) < w_s - \iota\right] = \exp\left\{\frac{1}{2}\varrho^2\right\} \varphi\left(\frac{w_s-\iota}{\varrho} - \varrho\right)
\end{aligned}$$

and, consequently

$$h_s(\iota) = d\exp\left\{\frac{1}{2}\varrho^2 + \iota\right\}\varphi\left(\frac{w_s-\iota}{\varrho} - \varrho\right) + p(t-s)\left[1 - \varphi\left(\frac{w_s-\iota}{\varrho}\right)\right].$$

The only serious obstacle complicating our effort is the fact that r past values of I are needed to get I_t from L_t; to overcome this, we suggest to assume $r = t$ for $t \le r$. Given this assumption, we easily get I from L by means of (5).

Step 3 – Econometrics of the common factors

Now that we have the values of the common factor, we may perform their estimation and eventual forecasting: by using the mappings (1) and (4), we may do the same for the pair Q, L. Note that it follows from our assumptions that the independence of Y and I implies the independence of Q and L; however, we do not expect the common factors to be independent in reality.

6. Conclusion

In the present paper, we suggested a potentially estimable model of credit losses. Even if it is rather general, a bit less could be assumed if a user wished it, especially

- the distribution of the individual factors need not be the same at all periods but it might depend on the time and on the past of the common factor,
- a dependence of the the individual factors ΔE_t^i and ΔZ_t^i could be established.

While the first generalization would not change our formulas much (some indices would have to be added to the present notation), the second one would bring the necessity to work with a conditional distribution of ΔE given not defaulting, for which no analytical formula exists even in the simple case of normal factors. Studying these and other generalization as well as application of the model in practice is a promising topic of a further research, as well as finding econometric explanations of the common factors.

Acknowledgements

This work is supported by the grants No. 402/09/0965 and No. 402/09/H045 of the Czech Science Foundation and by grant No. 46108 of Charles University (GAUK)

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